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The Generalization of Fractional Fourier-Cosine Transform

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ABSTRACT: Many Fourier Transform have been proposed, tested and proven efficient in solving many applications in various scientific fields. Apart from this, Fractional Fourier Cosine Transform comes from their outstanding ability to solve continuous variable rotational parameter, effectively analyzing signals in the time-frequency domain. It is highly effective for processing real-valued, even signals, reducing computational complexity in digital implementations compared to standard fractional techniques.

In this article, we are preparing a review research paper on Fractional Fourier-Cosine Transform.

KEYWORDS: Fractional Fourier transform, fractional Cosine transform, fractional Sine transform, continuous functions, signal processing, image processing, computational techniques.

I. INTRODUCTION

This review paper is based on the following research papers given in the table:

Sr. No.	Reference Research Articles Title	Articles by
1	Fourier Transform	Terence Tao
2	Fractional Cosine and Sine Transform in Relation to the Fractional Fourier and Hartley Transform	Tatiana Alieva Martin J. Bastiaans
3	Some Examples on Generalized Two-Dimensional Fractional Cosine Transform In the Range $-\infty$ to ∞	V. D. Sharma S. A. Khapekar
4	Mathematical Foundations and Applications of Fourier Transforms in Continuous Function	Vasanthakumari T N

[1] Based on the review of the information presented in the first research article “Fourier Transform” by “Tarence Tao”, Very broadly speaking, the Fourier transform is a systematic way to decompose “generic” functions into a superposition of “symmetric” functions. These symmetric functions are usually quite explicit (such as a trigonometric function $\sin(nx)$ or $\cos(nx)$), and are often associated with physical concepts such as frequency or energy. What “symmetric” means here will be left vague, but it will usually be associated with some sort of group G , which is usually (though not always) abelian. Indeed, the Fourier transform is a fundamental tool in the study of groups (and more precisely in the representation theory of groups, which roughly speaking describes how a group can define a notion of symmetry). The Fourier transform is also related to topics in linear algebra, such as the representation of a vector as linear combinations of an orthonormal basis, or as linear combinations of eigenvectors of a matrix (or a linear operator).

To give a very simple prototype of the Fourier transform, consider a real-valued function $f : R \rightarrow R$. Recall that such a function $f(x)$ is even if $f(-x) = f(x)$ for all $x \in R$, and is odd if $f(-x) = -f(x)$ for all $x \in R$. The Fourier transform decomposes a function exactly into many components, each of which has a precise frequency. In some applications it is more useful to adopt a “fuzzier” approach, in which a function is decomposed into fewer components, and each component has a range of frequencies rather than consisting purely of a single frequency. Such

decompositions can have the advantage of being less constrained by the uncertainty principle, which asserts that a strong localization in the frequency variable must be accompanied by a strong delocalization in the position variable; this leads to some variants of the Fourier transform, such as wavelet transforms, which are better suited for a number of problems in applied and computational mathematics, and also in certain questions in harmonic analysis and differential equations. The uncertainty principle, being fundamental to quantum mechanics, also connects the Fourier transform to mathematical physics, and in particular to the connections between classical and quantum physics, which can be studied using the methods of geometric quantization and microlocal analysis.

The fractional FT $F^\alpha(u)$ of a function $f(x)$ is defined as

$$F^\alpha(u) = R^\alpha F[f(x)](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\alpha(x, u) f(x) \exp(-jux / \sin \alpha) dx,$$

where

$$k_\alpha(x, u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j \sin \alpha}} \exp[\frac{1}{2}j(x^2 + u^2) \cot \alpha]$$

Note that for $\alpha = \frac{1}{2}\pi$, for which $k_{\pi/2}(x, u) = 1$.

[2] FRACTIONAL COSINE AND SINE TRANSFORMS-

Based on the review of the information presented in the second research article “Fractional Cosine and Sine Transform in Relation to the Fractional Fourier and Hartley Transform” by “Tatiana Alieva and Martin J. Bastiaans”, we have prepared the following equations which are helps to understand cosine and sine transforms.

We now restrict ourselves to a one-sided function $f(x)$, with $f(x) = 0$ for $x < 0$, and define the fractional CT and ST as

$$\begin{aligned} F_C^\alpha(u) &= R_C^\alpha[f(x)](u) \\ &= R_F^\alpha[f(x) + f(-x)](u) \\ &= F^\alpha(u) + F^\alpha(-u) \quad (u \geq 0) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty k_\alpha(x, u) f(x) \cos(ux / \sin \alpha) dx, \\ F_S^\alpha(u) &= R_S^\alpha[f(x)](u) \\ &= \exp(j\alpha) R_F^\alpha[f(x) - f(-x)](u) \\ &= \exp(j\alpha) [F^\alpha(u) - F^\alpha(-u)] \quad (u \geq 0) \\ &= -j \exp(j\alpha) \sqrt{\frac{2}{\pi}} \int_0^\infty k_\alpha(x, u) f(x) \sin(ux / \sin \alpha) dx \end{aligned}$$

respectively, which reduce to the normal CT and ST for $\alpha = \frac{1}{2}\pi$. To express, in a different way, the relationship between the fractional FT of a causal, one-sided function and the fractional CT and ST of this function, we can write

$$2F^\alpha(\pm u) = F_C^\alpha(u) \pm \exp(-j\alpha) F_S^\alpha(u).$$

Considering the kernels of the fractional transforms R^α ,

$$\begin{aligned} R_F^\alpha &: (1/\sqrt{2\pi}) k_\alpha(x, u) \exp(-jux / \sin \alpha), \\ R_C^\alpha &: (2/\sqrt{2\pi}) k_\alpha(x, u) \cos(ux / \sin \alpha), \\ j \exp(-j\alpha) R_C^\alpha &: (2/\sqrt{2\pi}) k_\alpha(x, u) \sin(ux / \sin \alpha), \end{aligned}$$

we can say that R_C^α is related to the even part of R_F^α , while R_S^α is related to the odd part of it. In general we conclude that, to determine the fractional CT of a causal, one-sided function $f(x)$, we can as well determine the fractional FT of

the evenly extended two-sided function $f(x) + f(-x)$; similarly, to determine the fractional ST of such a function, we can as well determine the fractional FT of the oddly extended two-sided function $\exp(j\alpha)[f(x) - f(-x)]$. And in both cases we restrict ourselves to $u \geq 0$.

[3] Examples on Generalized Two-Dimensional Fractional Cosine Transform-

Based on the review of the information presented in the third research article “Some Examples on Generalized Two-Dimensional Fractional Cosine Transform In the Range $-\infty$ to ∞ ” by “V. D. Sharma and S. A. Khapre”, we have prepared the following examples.

If $F_C^\alpha\{f(x, y)\}(u, v)$ denotes generalized two dimensional fractional Cosine transform of $f(x, y)$ then $F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\} = K_C^\alpha(a, b, u, v)$

Solution:

$$F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\} = \sqrt{\frac{1 - i \cot \alpha}{2\pi}} e^{\frac{i(x^2 + y^2 + u^2 + v^2) \cot \alpha}{2}} \int_{-\infty}^{\infty} \cos(\text{cosec} \alpha \cdot ux) dx \int_{-\infty}^{\infty} \cos(\text{cosec} \alpha \cdot vy) (\delta(x - a) \cdot \delta(y - b)) dy$$

$$\text{Let } A = \sqrt{\frac{1 - i \cot \alpha}{2\pi}}, B = e^{\frac{i(u^2 + v^2) \cot \alpha}{2}}$$

$$F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\} = AB \int_{-\infty}^{\infty} \delta(x - a) e^{\frac{i(x^2) \cot \alpha}{2}} \cos(\text{cosec} \alpha \cdot ux) dx$$

$$\int_{-\infty}^{\infty} \delta(y - b) e^{\frac{i(y^2) \cot \alpha}{2}} \cos(\text{cosec} \alpha \cdot uy) dy$$

$$F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\} = AB e^{\frac{i(a^2) \cot \alpha}{2}} \cos(\text{cosec} \alpha \cdot ua) e^{\frac{i(b^2) \cot \alpha}{2}} \cos(\text{cosec} \alpha \cdot vb)$$

We know that,

$$\int_{-\infty}^{\infty} \delta(t - a) \varphi(t) dt = \varphi(a) F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\}$$

$$= AB e^{\frac{i(a^2 + b^2) \cot \alpha}{2}} \cos(\text{cosec} \alpha \cdot ua) \cos(\text{cosec} \alpha \cdot vb)$$

$$= K_C^\alpha(a, b, u, v)$$

$$F_C^\alpha\{(\delta(x - a) \cdot \delta(y - b))\} = K_C^\alpha(a, b, u, v)$$

[4] APPLICATIONS-

Based on the review of the information presented in the last research article “Mathematical Foundations and Applications of Fourier Transforms in Continuous Function” by “Vasanthakumari T N”, we have prepared the following applications.

1) Signal Processing

i) Filtering and Signal Reconstruction

In signal processing, Fourier transforms are extensively used for filtering and reconstructing signals. For instance, consider a signal $x(t)$ with its Fourier transform $X(\omega)$. A filter $H(\omega)$ can be applied in the frequency domain to filter specific frequency components:

$$Y(\omega) = H(\omega)X(\omega)$$

The filtered signal (t) is then obtained by taking the inverse Fourier transform of $Y(\omega)$:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega$$

Filtering is essential for tasks such as removing high-frequency noise or isolating certain frequency bands (Oppenheim & Schaffer, 2010).

ii) Noise Reduction

Noise reduction involves separating the desired signal $s(t)$ from noise $n(t)$. Suppose $x(t) = s(t) + n(t)$. In the frequency domain, this is represented as:

$$X(\omega) = S(\omega) + N(\omega)$$

A low-pass filter $H(\omega)$ can be designed to suppress the noise component (ω):

$$Y(\omega) = H(\omega)X(\omega)$$

After filtering, the inverse Fourier transform yields the noise-reduced signal:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega$$

This technique is particularly useful in audio signal processing (Proakis & Manolakis, 2007).

2)Image Processing**i) Image Filtering and Enhancement**

Fourier transforms are vital in image processing for tasks like filtering and enhancement. An image $f(x,y)$ can be transformed to the frequency domain:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i(ux+vy)} dx dy$$

Applying a filter $H(u,v)$ in the frequency domain:

$$G(u,v) = H(u,v)F(u,v)$$

The filtered image is then obtained via the inverse Fourier transform:

$$g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u,v) e^{i(ux+vy)} du dv$$

This process can enhance features such as edges and textures (Gonzalez & Woods, 2018).

3)Medical Imaging: MRI Scan Analysis

Scenario: A hospital uses MRI scans to detect abnormalities in brain tissues. The MRI machine captures time domain signals representing the magnetic response of brain tissues. To analyze these signals, the hospital's radiologists use Fourier transforms.

The time-domain signal from the MRI is $s(t)$. The radiologist applies the Fourier transform to convert it to the frequency domain:

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt$$

In the frequency domain, $S(\omega)$ reveals distinct peaks corresponding to different tissue types and potential abnormalities. The radiologist uses this information to identify and diagnose conditions like tumors or hemorrhages.

4)Telecommunications: Signal Modulation

Scenario: A telecommunications company wants to transmit a voice signal $v(t)$ over a long distance. They use amplitude modulation (AM) to encode the signal onto a carrier wave for efficient transmission.

The voice signal $v(t)$ modulates a carrier wave $\cos(\omega_c t)$, producing the modulated signal:

$$s(t) = v(t)\cos(\omega_c t)$$

To analyze the modulated signal, the company uses the Fourier transform. The modulated signal in the frequency domain is:

$$s(\omega) = \frac{1}{2} [V(\omega - \omega_c) + V(\omega + \omega_c)]$$

This representation shows the signal's spectrum shifted by the carrier frequency ω_c , making it suitable for transmission.

II. CONCLUSION

This study delves into the fundamental concepts, definitions, examples, applications of Fractional Fourier Cosine Transforms based on the review of given reference research articles. We have introduced the fractional cosine and sine transforms, which are additive with respect to the parameter α and which are closely related to the fractional FT. The main properties and possible application of these transforms were discussed.

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